Binomial approximation to Black-Scholes price

Math 485

December 6, 2013

1 Goal:

To approximate the price obtained by Black-Scholes formula with the Bionmial tree model, where S_t is a geometric BM:

$$dS_t = rS_t dt + \sigma S_t dB_t \tag{1}$$

2 The Black-Scholes price:

For simplicty, we let $r = 0, \sigma = 0.1, T = 1, S_0 = 1000$ and $K = 1000$. Then the Black-Scholes formula for Euro-Call is

$$V_0 = S_0 N(d_1) - KN(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\frac{1}{2}\sigma^2 T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}} \\ &= 0.05 \end{aligned}$$

Similarly $d_2 = -0.05$. Thus $V_0 = 1000(0.52 - 0.48) = 40$.

3 The approximation:

We now divide $[0, 1]$ into $n = 5$ intervals. The discrete approximation to (1) is

$$S_{t_{k+1}} - S_{t_k} = rS_k(t_{k+1} - t_k) + \sigma S_k(B_{t_{k+1}} - B_{t_k})$$

where $k = 0, 1, ..., 5$ and $t_0 = 0, t_1 = 0.2, ..., t_4 = 0.8, t_5 = 1$.
$B_{t_{k+1}} - B_{t_k}$ has distribution $N(0, t_{k+1} - t_k)$. We approximate this by $\sqrt{t_{k+1} - t_k} Y_k$ where

$$\begin{aligned} Y_k &= 1 \text{ with probability } \frac{1}{2} \\ &= -1 \text{ with probability } \frac{1}{2}. \end{aligned}$$

For short-hand, we will write S_k for S_{t_k}. The evolution equation for S_k becomes

$$S_{k+1} = S_k(1 + \sigma\sqrt{t_{k+1} - t_k} Y_k).$$

Note that this is exactly the Binomial model we have studied before with
$X_k = 1 + \sigma\sqrt{t_{k+1} - t_k} Y_k$. Plug in , we have

$$\begin{aligned} X_k &= 1.044 \text{ with probability } \frac{1}{2} \\ &= .956 \text{ with probability } \frac{1}{2}. \end{aligned}$$

Draw out the binomial tree, we see that the price for Euro Call on S_k with strike 1000 and expiration time $n = 5$ is

$$V_0^b = \left(240 + 5 \times 135 + 10 \times 39.9\right) \frac{1}{2^5} = 41.06$$

This is not a very precise approximation to the Black-Scholes price of course (which gives 40 as in Section 2) but considering we only used 5 steps it is not terrible. The point of this computation is to convince you again that indeed the Geometric Brownian motion can be viewed as the limit of the Binomial tree as the time step gets closer to 0.